Uniform Interpolation for BPL

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Abstract

In this talk, we give a proof of uniform interpolation (UI) for Visser's Basic Propositional logic (BPL), a sub-logic of Intuitionist Propositional Logic (IPL) by using some techniques developed by Visser in [V$^+$96].

1 Introduction

We say logic L has Craig Interpolation property if $\mathsf{L} \vdash \phi \to \psi$ implies existence of $\chi(\vec{p})$ such that $\vec{p}$ is subset of intersection of atoms of ϕ and ψ and $\mathsf{L} \vdash \phi \to \chi$ and $\mathsf{L} \vdash \chi \to \psi$.

The uniform interpolation property is, in a sense, the generalization of Craig interpolation property. If instead of two formulas, we restrict the interpolant to a formula and a subset of its propositional variables (which are to be the shared variables), we reach a stronger definition: a uniform right-interpolant for $\phi(\vec{q}, \vec{p})$ with respect to $\vec{p}$ is a formula $\chi(\vec{p})$ such that for all formulas $\psi(\vec{p}, \vec{r})$ with $\mathcal{L} \vdash \phi \to \psi$, χ acts as an interpolant for ϕ and ψ. The uniform left-interpolant is defined analogously. A logic whose formulas have both left and right interpolants is said to satisfy the uniform interpolation property.

Uniform interpolation property for IPL was first proved by Pitts using syntactical methods [Pit92].The same results was proved using semantical methods independently in [V$^+$96] & [GZ95].

In this lecture we want to establish some new interpolational results on BPL and FPL using bisimulation techniques of [V$^+$96]. Basic Propositional Logic

BPL and Formal Propositional Logic FPL, both introduced by Visser, are logics that correspond to the modal counterparts $\mathsf{K4}$ and GL, respectively, in the same way that IPL corresponds to $\mathsf{S4}$. We can visualize it by the following informal equation:

$$\frac{\mathsf{IPL}}{\mathsf{S4}} = \frac{\mathsf{BPL}}{\mathsf{K4}} = \frac{\mathsf{FPL}}{\mathsf{GL}}$$

Model theoretically, we can view BPL as the logic of transitive but not necessarily reflexive Kripke frames. The Kripke frames of FPL in addition are Noetherian. In this paper, we use alternative sound and complete class of frames for these logics depending on our needs.

Let $\mathcal{L}$ be the language of Intuitionistic Propositional Logic IPL consisting of PV, a fixed set of propositional variables $p_0, p_1, \ldots$, propositional connectives $\wedge, \vee, \rightarrow$, and a propositional constant $\bot$. We assume that $p, q, r, \ldots$ range over propositional variables, $\phi, \psi, \chi, \ldots$ range over arbitrary formulas, and $\vec{p}, \vec{q}, \vec{r}, \ldots$ range over *finite* sets of propositional variables. For $\vec{p}$ and $\vec{q}$, we abbreviate $\vec{p} \cup \vec{q}$ by $\vec{p}, \vec{q}$. $\neg\phi$ is defined as $\phi \rightarrow \bot$, $\phi \leftrightarrow \psi$ is defined as $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, and $\top$ is defined as $\neg\bot$. PV together with $\top$ and $\bot$ is the set of *atoms*. For a set of propositional variables $\mathcal{P}$, we write $\mathcal{L}(\mathcal{P})$ for $\mathcal{L}$ restricted to $\mathcal{P}$. Similar notations will be used for other classes of formulas. We also denote $\mathsf{FPL} \vdash \phi$ and $\mathsf{BPL} \vdash \phi$ by $\vdash \phi$, when clear from the context.

Definition 1. A *Kripke frame* is structure $\mathfrak{F} = \langle W, \prec \rangle$, where $W \neq \varnothing$, and $\prec$ is a binary relation on W.

Definition 2. A *Kripke model* is a structure $\mathfrak{M} = \langle \mathfrak{F}, \ \mathcal{P}, \ \Vdash \rangle$, where $\mathfrak{F}$ is a Kripke frame, $\mathcal{P}$ is a set of propositional variables, and $\Vdash$ is the atomic forcing relation on $\mathcal{P}$. Forcing should satisfy the following condition:

$$m \prec m' \text{ and } m \Vdash p \Rightarrow m' \Vdash p.$$

We say $\mathfrak{M} \Vdash p$ when $\forall m \in W \ m \Vdash p$, or equivalently $r \Vdash p$. For $\phi \in \mathcal{L}(\mathcal{P})$, $\mathfrak{M} \Vdash \phi$ is defined in the standard way (see e.g. [CZ97]).

if $m \in W$ we call $\langle \mathfrak{M}, m \rangle$ a pointed model and if in addition m is the least element of W with respect to $\prec$, call it a rooted model. class of all models, pointed models and rooted models respectively represented by Mod, Pmod and Rmod.

We will stick to the following characterization of BPL and FPL models throughout the section:

Theorem 1 (Kripke Completeness)**.**

1. BPL is complete with respect to transitive irreflexive models.
2. FPL is complete with respect to transitive irreflexive Noetherian models.

We extend the natural numbers ω with an extra element ∞. Let w^{∞} be $\omega \cup \{\infty\}$ with the obvious ordering $\leq$, addition and subtraction. We let n range over ω, and α range over ω^{∞}.

Next, we recall the notion of bisimulation (and in general, layered bisimulation) between two models. For convenience in our context, this notion has been slightly modified, i.e. the forth and back conditions hold strictly.

Definition 3. Consider $\mathcal{P}$-models $\mathfrak{M}$ and $\mathfrak{N}$. A *layered bisimulation* or simply *l-bisimulation* Z between $\mathfrak{M}$ and $\mathfrak{N}$ is a ternary relation between $\mathfrak{M}$, ω^∞ and $\mathfrak{N}$, satisfying the conditions specified below. We will also consider Z as an ω^∞-indexed set of binary relations between $\mathfrak{M}$ and $\mathfrak{N}$ writing $mZ_\alpha n$ for $\langle m, \alpha, n\rangle \in Z$. We often write kZm for $mZ_\infty n$. We give the conditions:

- $mZ_\alpha n \Rightarrow PV(m) = PV(n)$.
- (forth) $m' \succ mZ_{\alpha+1}n \Rightarrow$ there exists n' such that $m'Z_\alpha n' \succ m$.
- (back) $mZ_{\alpha+1}n \prec n' \Rightarrow$ there exists m' such that $m \prec m'Z_\alpha n'$.

for pointed models $\langle\mathfrak{M}, m\rangle$ and $\langle\mathfrak{N}, n\rangle$ if we have $mZ_\alpha n$ we say $\mathfrak{M} \simeq_\alpha \mathfrak{N}$.

Next, we'll define the complexity measure i.

Definition 4. $i : \mathcal{L}(\mathcal{P}) \to \omega$ recursively defined by:

- $i(p) := i(\bot) := i(\top) := 0$,
- $i(\phi \wedge \psi) := i(\phi \vee \psi) := max(i(\phi), i(\psi))$,
- $i(\phi \to \psi) := max(i(\phi), i(\psi)) + 1$.

We also let:

Definition 5.

- $\mathcal{L}_n(\mathcal{P}) := \{\phi \in \mathcal{L}(\mathcal{P}) \mid i(\phi) \leq n\}$,
- $\mathcal{L}_\infty(\mathcal{P}) := \mathcal{L}(\mathcal{P})$.

2 Main Results

Next theorem show that formulas with complexity at most n, act "nicely" under layered n-bisimulation. first we need a definition.

Definition 6.

- Let $\langle\mathfrak{M}, m\rangle$ be transitive irreflexive pointed models then $Th_{\mathsf{B}_\alpha}(\mathfrak{M}) = \{\phi | m \Vdash_{\mathsf{BPL}} \phi\}$.
- Let $\langle\mathfrak{M}, m\rangle$ be transitive irreflexive Noetherian pointed models then $Th_{\mathsf{F}_\alpha}(\mathfrak{M}) = \{\phi | m \Vdash_{\mathsf{FPL}} \phi\}$.

Theorem 2 (Bisimulation)**.**

1. Let $\langle \mathfrak{M}, m\rangle$ and $\langle \mathfrak{N}, n\rangle$ be transitive irreflexive pointed models with $\mathfrak{M} \simeq_\alpha \mathfrak{N}$. We have $Th_{\mathsf{B}_\alpha}(\mathfrak{M}) = Th_{\mathsf{B}_\alpha}(\mathfrak{N})$.
2. Let $\mathfrak{M}$ and $\mathfrak{N}$ be transitive irreflexive Noetherian models with $\mathfrak{M} \simeq_\alpha \mathfrak{N}$. We have $Th_{\mathsf{F}_\alpha}(\mathfrak{M}) = Th_{\mathsf{F}_\alpha}(\mathfrak{N})$.

Theorem 3 (Amalgamation)**.** Consider disjoint propositional variables $\vec{q}, \vec{p}, \vec{r}$. Let $X \subseteq \mathcal{L}(\vec{q}, \vec{p})$ be a finite set closed under subformula. Let $\langle \mathfrak{M}, m\rangle \in \mathrm{Pmod}(\vec{q}, \vec{p})$, $\langle \mathfrak{N}, n\rangle \in \mathrm{Pmod}(\vec{p}, \vec{r})$. Let:

$$v := \mid \{c \in X \mid c \text{ is a propositional variable or an implication}\} \mid .$$

If $m_0(\vec{p}) \simeq_{2v+1} n_0(\vec{p})$, Then there exists a $\vec{q}, \vec{p}, \vec{r}$-model $\langle \mathfrak{K}, k_0\rangle$ such that $k_0(\vec{p}, \vec{r}) \simeq n_0$ and $Th_X(k_0) = Th_X(m_0)$.

Finally we prove Uniform Interpolation for FPL and FPL using Amalgamation theorem.

Theorem 4 (Uniform Interpolation)**.** BPL and FPL satisfies the uniform interpolation property.

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